The Theoretical Minimum

Special Relativity and Classical Field Theory - Solutions

L01E00

Last version: tales.mbivert.com/on-the-theoretical-minimum-solutions/ or github.com/mbivert/ttm

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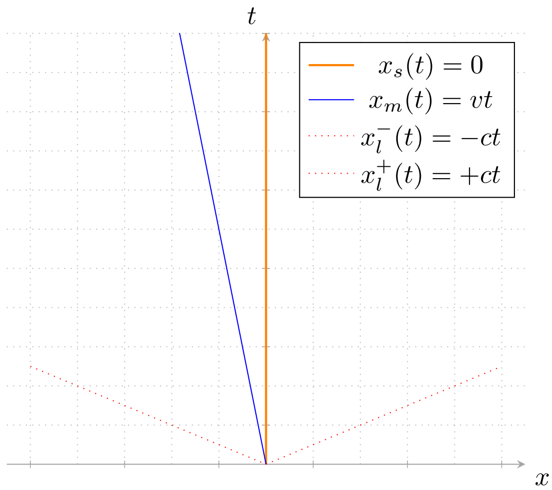
Exercise 1 (p. 8)**.** [. . .] *As an exercise you can redraw them for negative v.*

This is an "optional" exercise from section 1.2: we're considering two reference frames, one static (ours) and one moving at velocity v, in a one-spatial dimension setting. In the graph below:

- $x_s(t)$ (s for static) represents the coordinates over time of our fixed reference frame;
- $x_m(t)$ (m for moving) represents the coordinates over time of the moving reference frame; we're assuming the velocity to be oriented negatively (i.e. $v < 0$)
- $x_l^{\pm}(t)$ (l for light) represents the coordinates over time of a light ray being emited to the left/right, in the (unique) spatial direction.

Where all those position functions are given in the *static frame of reference.*



Remark 1. *Note that this is just the mirror image of what we would have with positive velolicities (reflection about the vertical t axis).*

Remark 2. *I'll do my best to use an unambiguous notation. The authors for instance in their graphic* 1. *use the same symbol "x" to denote three distinct things:*

- *Two position functions ($x_l^+(t)$ and $x_m(t)$)*
- *The spatial coordinate of an arbitrary, punctual event.*

Exercise 2 (p. 10)**.** [. . .] *Can we invert the relationship? That's easy and I will leave it to you.*

This is a second optional exercise, a little bit later in the same 1.2 section, in the same setting of two frames, one static, one moving at a constant velocity (speed of v).

Suppose an event $(x_0, t_0)_s$ happens, where I've used the s subscript to indicate that the coordinates are to be understood within the *static* frame of reference. The authors just showed us how we could express the coordinates of this event in the moving reference frame, say $(x'_0, t'_0)_m$, where the m subscript indicates that the coordinates are expressed in the *moving* reference frame.

And we're asked to do the thing in the reverse order. So, assume we're given an event of coordinate $(x'_0, t'_0)_m$ in the moving frame. We're looking to express its coordinates $(x_0, t_0)_s$ in the static frame in terms of $(x'_0, t'_0)_m$:

$$x_0 = f(x'_0, t'_0); \qquad t_0 = g(x'_0, t'_0)$$

By assumption, time flows equally in all reference frames[1], hence:

$$\boxed{t_0 = t'_0}$$

Now if an event happens at position x'_0 at a given time ($t'_0 = t_0$) in the moving frame of reference, to express it in the static frame of reference, we need to take into account the relative motion between the two frames. As one of them is static, this relative motion is solely equivalent to the motion of the moving frame. Furthermore, there's only one spatial dimension, so we just need to shift the position by how much the moving frame has moved at the given time $t'_0 = t_0$:

$$x_0 = x'_0 + x_m(t'_0) \qquad \Leftrightarrow \qquad \boxed{x_0 = x'_0 + vt'_0}$$

How do we know this should be a + and not a − thought? Well, because this is a very simple case, we can take some shortcuts, but it might help to consider things a bit more generally: the moving frame moves at some velocity $\boldsymbol{v}$, whose spatial coordinates expressed in the static frame are:

$$\boldsymbol{v} = \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix}_s \qquad \text{with } v > 0$$

Note however that from the point of view of the moving reference frame, the static frame "moves" with a velocity $\boldsymbol{v}'$:

$$\boldsymbol{v}' = \begin{pmatrix} -v \\ 0 \\ 0 \end{pmatrix}_m \qquad \text{still with } v > 0$$

This speed now has a *negative* velocitity in the x-axis, because the "static" frame is apparently "moving" away to the left of the "moving" frame.

That's to say, the previous + sign is "arbitrary" somehow, it's just a way to take into account the effect of this relative motion.

One more thought: this is true for any event, so we could use x and t instead of x_0 and t_0 to signify that it's true on the full space instead of at a peculiar point. It's true in particular for all points describing the motion of the ray of light $x_l^+(t)$:

$$ct =: x_l^+(t) = x' + vt \qquad \Leftrightarrow \qquad x' = (c - v)t = (c - v)t'$$

[1] In *this* very peculiar context